Substitution Rule (Section 5.5) and Area between Curves (Section 6.1)

Substitution Rules

If $u = g(x)$ is a differentiable function, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Key Idea: We're looking for a factor of the integrand which looks like the derivative of another factor of the integrand

Example

$$\int \sin(x^2)2x\,dx$$

Identify what to use for u

General principles for identifying u:

- Look for one part of the function whose derivative shows up elsewhere in the function
- Look for a substitution to make the problem simpler
- Look for a substitution to let us use a known integration rule

Example

Use a substitution to solve the integral

$$\int (5x^2 + 2)^{3/2} x \, dx$$

Think: Chain Rule Backwards

Chain Rule:

$$\sin(x^3) = \cos(x^3) \cdot 3x^2$$

Integrate using substitution:

$$\int \cos(x^3) \cdot 3x^2 \, dx = \int \cos(u) \, du = \sin(x^3) + C$$

Try it!

$$\int \left(\frac{x}{3}+2\right)^{1/2}\,dx$$

Example

$$\int \sin(x)^2 \cos(x)\, dx$$

Try it!

$$\int \tan(x)\, dx$$

Example

$$\int \frac{\ln(x)^2}{x}\, dx$$

Substitution Rule for Definite Integrals

If $u = g(x)$ is a differentiable function, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example

$$\int_0^2 e^{x^2} x \, dx$$

Example

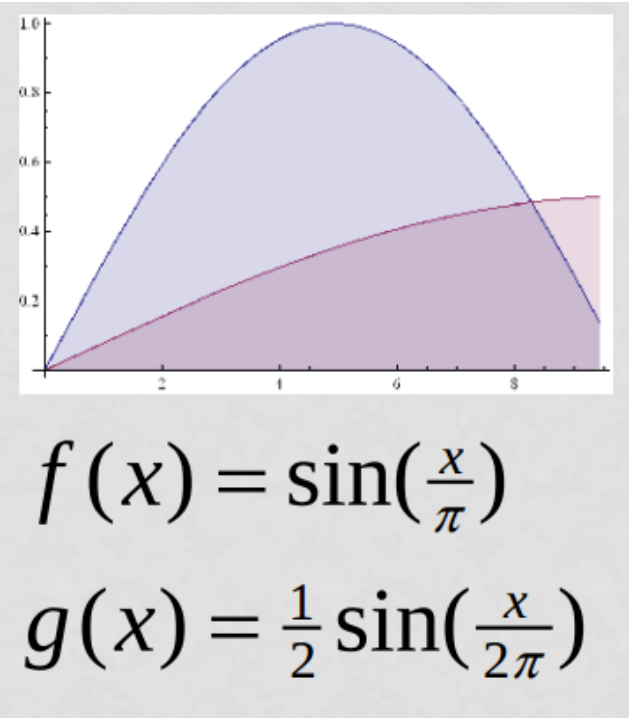
$$\int_3^5 \frac{1}{1-x}\,dx$$

Try it!

$$\int_2^4 \frac{x^2}{x^3 - 1}\, dx$$

Area between two curves

Find the area between the two curves in the region shown.



Any ideas?

Process

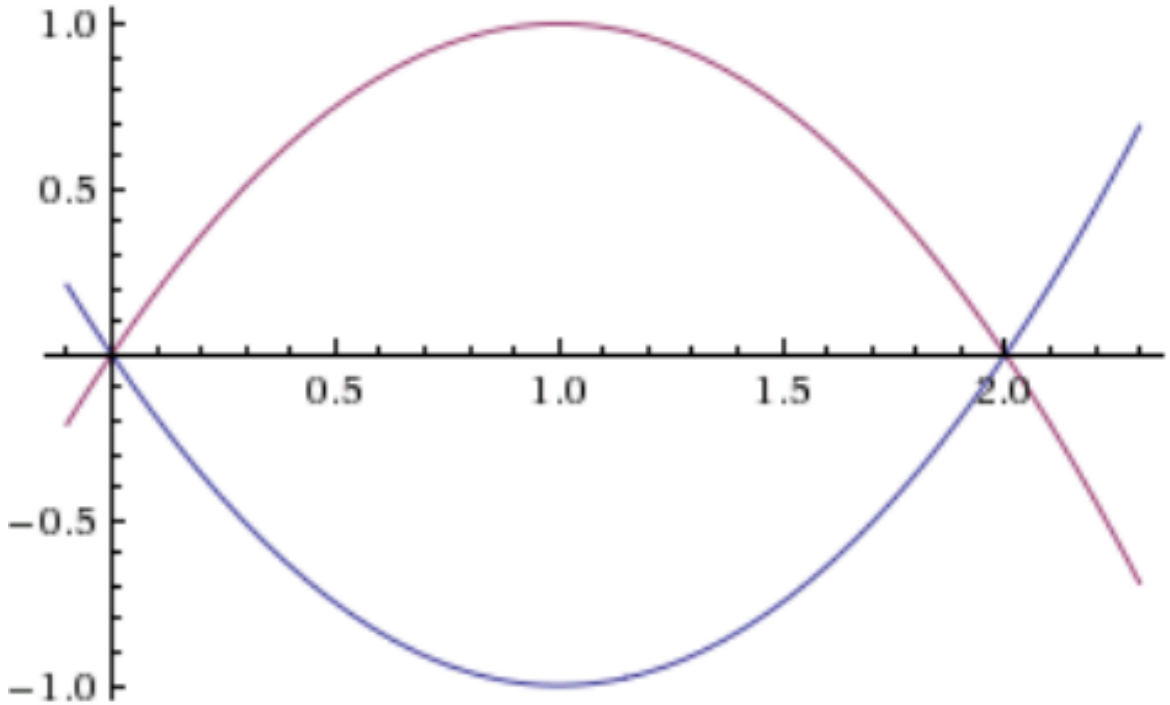
- The basic process is to compute

$$\int_a^b f(x) - g(x)\,dx$$

- We need to pay attention to which curve is on top
- We might need to calculate a and b separately
- Answer should always be positive

Negative Area?

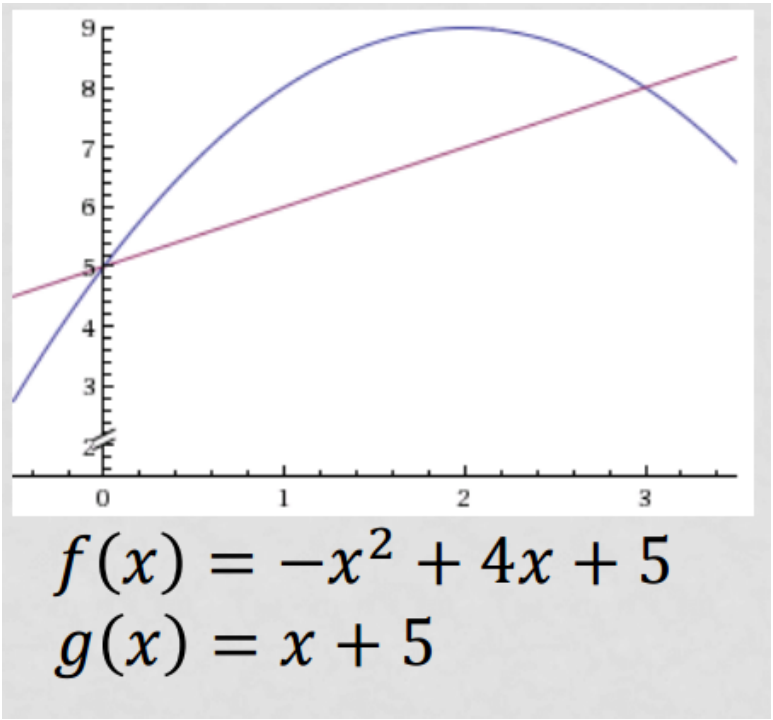
What if one of the curves is negative?



The formula takes care of the negative signs on its own.

Example

Find the area between the two curves in the region shown.



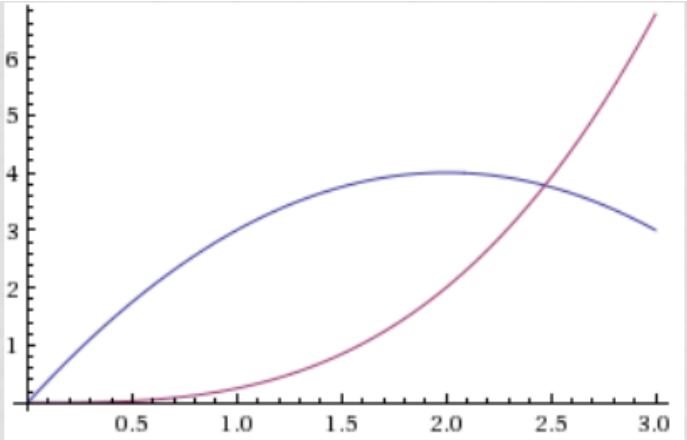
Steps

- Set $f(x) = g(x)$ and solve for x to get the limits of integration (a, b).
- Find out which curve is the top curve (plot them or plug in an x-value)
- Use the formula

$$\int_a^b f(x) - g(x)\, dx$$

Example

Find the area between the two curves in the region shown.

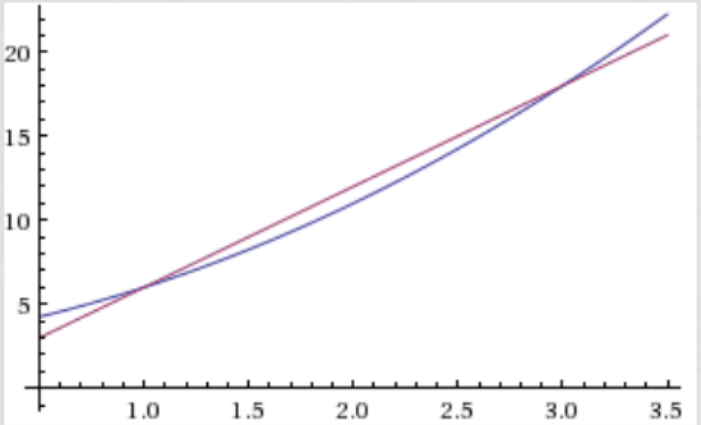


$$f(x) = -(x-2)^2 + 4$$

$$g(x) = \frac{1}{4}x^3$$

Try it!

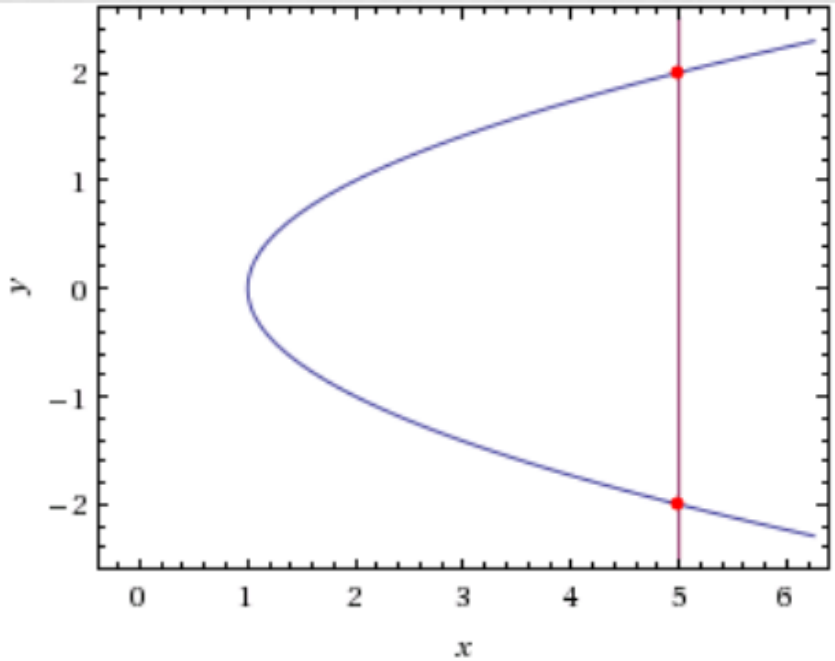
Find the area between the two curves in the region shown.



$$f(x) = x^2 + 2x + 3$$

$$g(x) = 6x$$

Example in terms of y



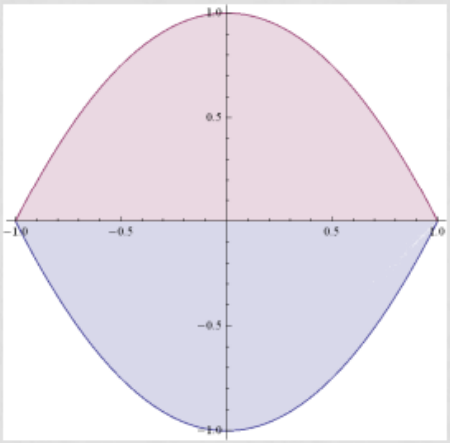
Find the area between the two curves.

$$y^2 = x - 1$$

$$x = 5$$

Try it!

What is the difference between this area and the area of a circle of radius 1?

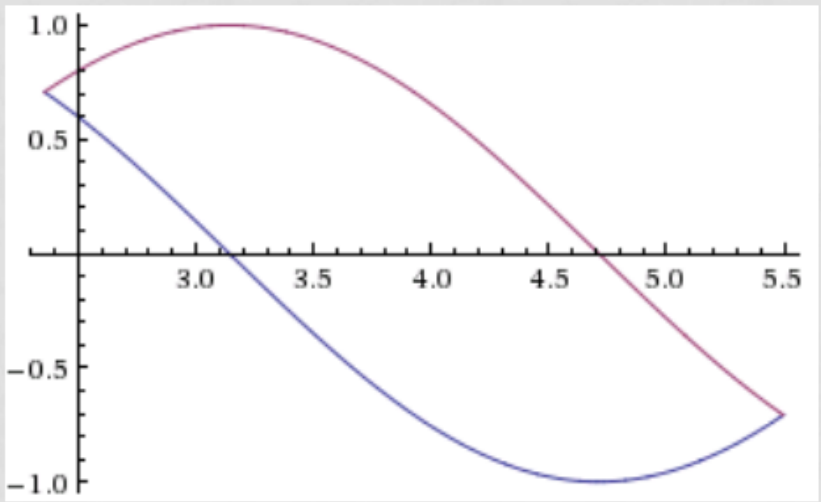


$$f(x) = x^2 - 1$$

$$g(x) = -x^2 + 1$$

Example

Find the area between the two curves in the region shown.



$$f(x) = \sin(x)$$
$$g(x) = \cos(x - \pi)$$