On metaplectic modular categories

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Motivation: The Property F conjecture

Conjecture (Rowell)

Let $\mathcal{C}$ be a braided fusion category and let X be a simple object in $\mathcal{C}$. The braid group representations $\mathcal{B}_n$ on $\mathrm{End}(X^{\otimes n})$ have finite image for all $n > 0$ if and only if X is weakly integral (i.e. $\mathrm{FPdim}(X)^2 \in \mathbf{Z}$).

- Verified for modular categories from quantum groups (Rowell, Naidu, Freedman, Larsen, Wang, Wenzl, Jones, Goldschmidt)

A potential approach to property F for modular categories

Prove two conjectures:

(1) Gauging Conjecture (Ardonne–Cheng–Rowell–Wang): Gauging preserves property F

(2) Weakly Group-theoretical Conjecture (Etingof–Nikshych–Ostrik): Every weakly integral modular category is weakly-group theoretical.

Theorem (Natale, 2017)

Every weakly group-theoretical modular category is a gauging of a pointed or pointed$\boxtimes$Ising MTC

Gauging a modular category

- Starting point: a group homomorphism $\rho : G \to \mathrm{Aut}_{\otimes}^{br}(\mathcal{B})$
- Extend $\mathcal{B}$ by ρ to get a G-crossed braided category $\mathcal{B}_G^{\times}$
  - Cohomological obstructions must vanish for the extension to exist
  - Choices for fusion rules, associators
- Equivariantize

Odd metaplectic modular categories

An odd metaplectic modular category is a unitary modular category with the same fusion rules as $SO(N)_2$ for some odd $N > 1$. It has 2 simple objects X_1, X_2 of dimension $\sqrt{N}$, two simple objects $1, Z$ of dimension 1, and $\frac{N-1}{2}$ objects Y_i, $i = 1, \ldots, \frac{N-1}{2}$ of dimension 2. All simple objects are self-dual.

The fusion rules are:

1. $Z \otimes Y_i \cong Y_i$, $Z \otimes X_i \cong X_{i+1}$ (modulo 2), $Z^{\otimes 2} \cong 1$,
2. $X_i^{\otimes 2} \cong 1 \oplus \bigoplus_i Y_i$,
3. $X_1 \otimes X_2 \cong Z \oplus \bigoplus_i Y_i$,
4. $Y_i \otimes Y_j \cong Y_{\min\{i+j, N-i-j\}} \oplus Y_{|i-j|}$, for $i \neq j$ and $Y_i^{\otimes 2} = 1 \oplus Z \oplus Y_{\min\{2i, N-2i\}}$.

Previous Work

Theorem (Rowell–Wenzl)

The images of the braid group representations on $\mathrm{End}_{SO(N)_2}(S^{\otimes n})$ for N odd are isomorphic to images of braid groups in Gaussian representations; in particular, they are finite groups.

Theorem (Ardonne–Cheng–Rowell–Wang, Bruillard–Plavnik–Rowell)

1. *Suppose $\mathcal{C}$ is a (not necessarily odd) metaplectic modular category with fusion rules $SO(N)_2$ with $4 \nmid N$, then $\mathcal{C}$ is a gauging of the particle-hole symmetry of a $\mathbb{Z}_N$-cyclic modular category.*
2. *For $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ with distinct odd primes p_i, there are exactly 2^{s+1} many inequivalent metaplectic modular categories.*

"Even-even" metaplectic modular categories

An "even-even" metaplectic modular category is a unitary modular category with the same fusion rules as $SO(2k)_2$ for even $k \geq 2$. It has 4 simple objects V_1, V_2, W_1, W_2 of dimension $\sqrt{k}$ and four simple objects $1, f, g, fg$ of dimension 1. Setting $r := \frac{k}{2} - 1$, there are $k-1$ objects X_i, $i = 0, \ldots, X_{r-1}$ and $Y_0, \ldots, Y_r$ of dimension 2. All simple objects are self-dual.

Even-even metaplectic modular categories

The key fusion rules are:

- $f^{\otimes 2} = g^{\otimes 2} = 1$, $f \otimes X_i = g \otimes X_i = X_{r-i-1}$ and $f \otimes Y_i = g \otimes Y_i = Y_{r-i}$
- $g \otimes V_1 = V_2, f \otimes V_1 = V_1$ and $f \otimes W_1 = W_2, g \otimes W_1 = W_1$
- $V_1^{\otimes 2} = 1 \oplus f \oplus \bigoplus_{i=0}^{r-1} X_i$
- $W_1^{\otimes 2} = 1 \oplus g \oplus \bigoplus_{i=0}^{r-1} X_i$
- $W_1 \otimes V_1 = \bigoplus_{i=0}^{r} Y_i$
- $X_i \otimes X_j = \begin{cases} X_{i+j+1} \oplus X_{j-i-1} & i < j \leq \frac{r-1}{2} \\ 1 \oplus fg \oplus X_{2i+1} & i = j < \frac{r-1}{2} \\ 1 \oplus f \oplus g \oplus fg & i = j = \frac{r-1}{2} < r-1 \end{cases}$
- $Y_i \otimes Y_j = \begin{cases} X_{i+j} \oplus X_{j-i-1} & i < j \leq \frac{r}{2} \\ 1 \oplus fg \oplus X_{2i} & i = j \leq \frac{r-1}{2} \\ 1 \oplus f \oplus g \oplus fg & i = j = \frac{r}{2}. \end{cases}$

Analogous theorem for even-even metaplectic modular categories

Theorem (Bruillard–G–Plavnik–Rowell)

If $\mathcal{C}$ is a metaplectic modular category with the fusion rules of $SO(N)_2$ with $4 \mid N$ then the de-equivariantization $\mathcal{D} := \mathcal{C}_{\mathbb{Z}_2}$ by $\langle fg \rangle = \mathrm{Rep}(\mathbb{Z}_2)$ is a generalized Tambara-Yamagami category of dimension $4N$, and, the trivial component $\mathcal{D}_0 := [\mathcal{C}_{\mathbb{Z}_2}]_0 \cong \mathcal{C}(\mathbb{Z}_N, q)$ is a pointed cyclic modular category. Moreover, $\mathcal{C}$ is obtained from $\mathcal{C}(\mathbb{Z}_N, q)$ via a $\mathbb{Z}_2$-gauging of the particle-hole symmetry.

Degenerate case: we have $SO(4)_2 = \mathit{Ising} \boxtimes \mathit{Ising}$.

Count for even-even metaplectic modular categories (BGPR)

- Suppose we have the prime factorization $N = 2^a p_1^{a_1} \cdots p_s^{a_s}$ for $a > 2$.
- Let $\rho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_N)$ denote the map determined by $\rho(1)(n) = -n$, i.e. the particle-hole symmetry.

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- There are two potential obstructions to extending $\mathcal{C}(\mathbb{Z}_N, q)$ by ρ.
  1. The first obstruction $H^3_\rho(\mathbb{Z}_2, \mathbb{Z}_N)$ vanishes since we know that a gauging exists.
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- The action of $H^2_\rho(\mathbb{Z}_2, \mathbb{Z}_N)$ on the fusion rules is trivial.
- There is a choice of associativity constraints on the $\mathbb{Z}_2$-extension, so that *a priori* we have 4 distinct theories.
- Labelling ambiguity reduces the factor of 4 to 3
- This gives $3(2^{s+2})$ metaplectic modular categories of dimension $4N > 16$ with $4 \mid N$.

Current work (jww Eric Rowell and Yuze Ruan)

- Sequential gauging for even metaplectics
- Property F for metaplectic modular categories

Thanks

Thanks for listening!