The Definite Integral (Section 5.2)

Intro

Today we'll introduce the integral. We also make the connection between the integral and the shape of a graph.

Overview

- Notation for the definite integral
- Estimating integrals using Riemann sums
- Calculating integrals using shapes
- Properties of integrals

Notation

The process of adding up smaller and smaller rectangles to calculate the area under the curve is called a "definite integral."
To calculate the area under $f(x)$ from $[a, b]$, we write

$$\int_a^b f(x)\,dx$$

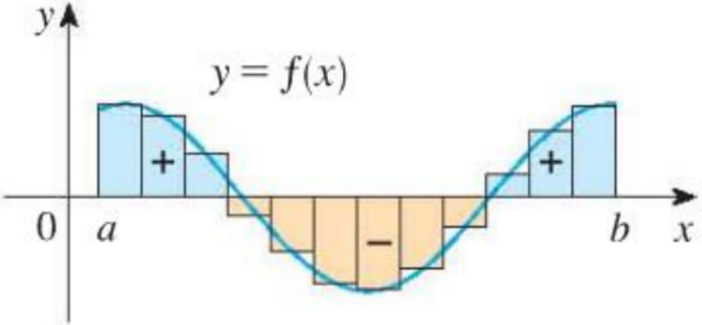
Example

Use a left Riemann sum with $n = 5$ to estimate the integral

$$\int_2^7 x + \frac{6}{x}\,dx$$

"Negative" Area

If there is area below the x-axis, we count it as negative in the integral.



"Negative" Area

So then

$$\int_a^b f(x)\,dx$$

means we add up all the area above the x-axis and subtract all the area below the x-axis.

Note: we we use antidifferentiation to compute integrals, this will take care of itself.

Integrals Using Shapes

Since we know that the integral equals the signed area between $f(x)$ and the x-axis, we can calculate integrals of functions whose graphs are simple shapes.

Example

Use the shape of the graph to evaluate

$$\int_{-4}^{8} \left(\frac{1}{2}x - 3\right)\, dx$$

Example

Calculate the integral:

$$\int_{-2}^{5} f(x)\, dx$$

For the function

$$f(x) = \begin{cases} \sqrt{4 - x^2}, & -2 < x < 0 \\ x + 2, & x \geq 0 \end{cases}$$

Try it!

Use the geometric shape of the graph to evaluate the integral

$$\int_{-3}^{2} (3x + 1)\, dx$$

Integral Rules

- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\displaystyle\int_a^a f(x)\,dx = 0$
- $\displaystyle\int_a^b \,dx = b - a$
- $\displaystyle\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$

Integral Rules

- $\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Question: Why is the last rule true?

Example

Use integral rules to write the sum as a single integral

$$\int_{-2}^{2} f(x)\,dx - \int_{5}^{2} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

Example

If we know that

$$\int_1^5 f(x)\,dx = 12$$

and

$$\int_4^5 f(x)\,dx = 4,$$

find

$$\int_1^4 2 \cdot f(x)\,dx$$

Example

6. The graph of g is shown. Estimate $\int_{-3}^{3} g(x)\,dx$ with six subintervals using (a) right endpoints, (b) left endpoints, and (c) midpoints.