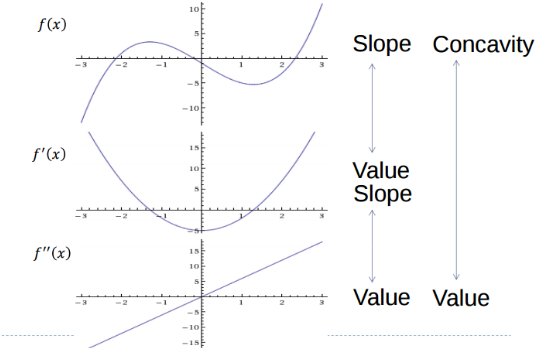
Optimization (Section 4.8) and Approximating Area (Section 5.1)

Intro

Today we'll start talking about integration, the opposite operation to taking a derivative. We will find the antiderivative of a function in a few easy cases.

Slope/Value Correspondences



Going backwards

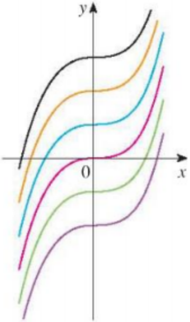
Question: Given a graph of $f'(x)$, can we draw a graph of the original function $f(x)$?

Going backwards: Example

$$f'(x) = 2x + 3$$

Going backwards

We have many choices of the antiderivative of $f(x)$ by moving the graph up and down.



Antiderivative

Given a function $f(x)$, we call $F(x)$ an antiderivative of $f(x)$ if

$$F'(x) = f(x)$$

Antiderivative

An antiderivative can always be moved up and down. So we will write *the* antiderivative as

$$F(x) = function + C,$$

where C denotes an unknown constant number. If we know that the antiderivative passes through a particular point, we can solve for C.

Antiderivative

Think of the antiderivative as being the opposite operation of the derivative, like $+/-$ or $*/\div$

Antiderivative - Rules

Using guess and check, can we find an antiderivative of $f(x) = x$?

Antiderivative - Rules

Antiderivative Power Rule for $f(x) = x^n$, as long as $n \neq -1$:

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

Notice that this is the opposite of the power rule for derivatives.

Antiderivative - Rules

In particular, this means that the antiderivative of $f(x) = 1$ is

$$F(x) = x$$

Example

Find the antiderivative of

$$f(x) = \frac{3x^3 + 2\sqrt{x}}{x^2}$$

Other antiderivatives

The antiderivative rules are just the derivative rules backwards:

$f(x)$	$F(x)$
x^n	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln\|x\| + C$
e^x	$e^x + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$

Warning!

The antiderivative of products and quotients works very differently.

Example

Find the antiderivative of $f(x)$:

$$f(x) = 5e^x + \frac{1}{x} + 3\sin(x)$$

Example

Find the antiderivative of $f(x)$:

$$f(x) = x^{3/4} + \sec^2(x) - \frac{10}{x}$$

Example

Find the exact antiderivative of $f'(x)$ if $f(-2) = 7$:

$$f'(x) = 3x^2 + 6x + 2$$

Example

Find $f(x)$ if

$$f''(x) = e^x + 10x$$

Finding f from f''

If you're given $f''(x)$ and asked for $f(x)$, you take two antiderivatives. This will give you two unknown constants C and D. If you're given two points on $f(x)$, then you can solve for both C and D.

Acceleration

Acceleration is he same thing as $f''(x)$. So if they give acceleration, take the antiderivative twice!

Acceleration

Acceleration on earth equals about $-10\,m/sec^2$. Suppose a ball is thrown in the air (from the ground) at $30\,m/sec$. Write the equation for the ball's position.

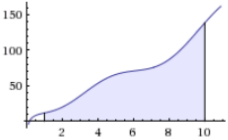
Approximating Area

Now we'll discuss how to approximate the area under a curve using rectangles. This leads to the definition of the integral. It turns out that integration involves taking antiderivatives.

Formulas for Area

$A = bw$

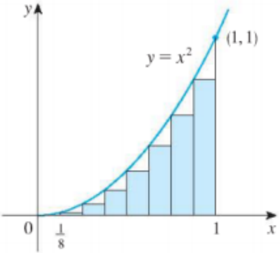
$A = \pi r^2$



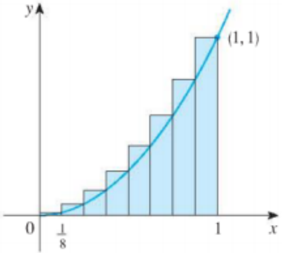
$A = ????$

Area under a Curve

We can get a rough estimate of area under a curve by breaking it into rectangles.



(a) Using left endpoints



(b) Using right endpoints

Left Endpoint Formula

- Break the interval $[a, b]$ up into n pieces
- Write down the endpoints of each piece as $x_0, x_1, x_2, \ldots, x_n$
- The formula for the left endpoint estimate is

$$\Delta x \cdot f(x_0) + \Delta x \cdot f(x_1) + \cdots + \Delta x \cdot f(x_{n-1})$$

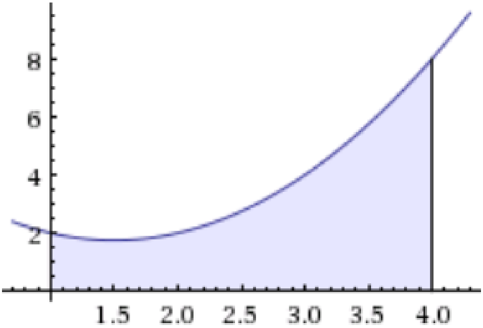
Left Endpoint Formula

- Break the interval $[a, b]$ up into n pieces
- Write down the endpoints of each piece as $x_0, x_1, x_2, \ldots, x_n$
- The formula for the left endpoint estimate is

$$= \frac{b-a}{n}(f(x_0) + f(x_1) + \cdots + f(x_{n-1}))$$

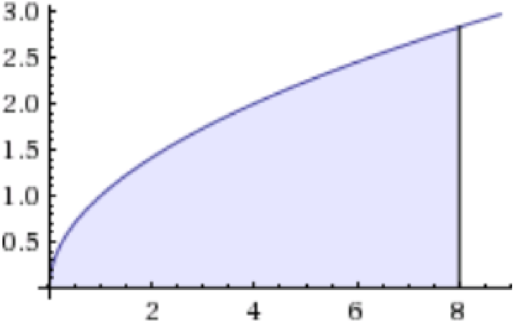
Example

Estimate the area under the curve $f(x) = x^2 - 3x + 4$ on the interval $[1, 4]$ using 3 left endpoint rectangles.



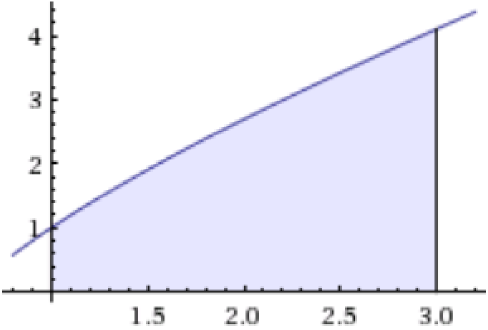
Example

Estimate the area under the curve $f(x) = \sqrt{x}$ on the interval $[0, 8]$ using 4 right endpoint rectangles.



Example

Estimate the area under the curve $f(x) = \ln(x) + x$ on the interval $[1, 3]$ using 4 midpoint rectangles.



HW Tip

On the car problem, try making a table of distance traveled in each time interval.